

Quantum algorithm for a generalized hidden shift problem

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ABSTRACT. Consider the following generalized hidden shift problem: given a function f on $\{0, \dots, M-1\} \times \mathbb{Z}_N$ satisfying $f(b, x) = f(b+1, x+s)$ for $b = 0, 1, \dots, M-2$, find the unknown shift $s \in \mathbb{Z}_N$. For $M = N$, this problem is an instance of the abelian hidden subgroup problem, which can be solved efficiently on a quantum computer, whereas for $M = 2$, it is equivalent to the dihedral hidden subgroup problem, for which no efficient algorithm is known. For any fixed positive ϵ , we give an efficient (i.e., $\text{poly}(\log N)$) quantum algorithm for this problem provided $M \geq N^\epsilon$. The algorithm is based on the “pretty good measurement” and uses H. Lenstra’s (classical) algorithm for integer programming as a subroutine.

1. Introduction

Quantum mechanical computers can solve certain problems asymptotically faster than classical computers, but the extent of this advantage is not well understood. The most significant example of quantum computational speedup, Shor’s algorithm for factoring and calculating discrete logarithms [26], is essentially based on an efficient quantum algorithm for the abelian hidden subgroup problem. This naturally leads to the question of whether the general *nonabelian* hidden subgroup problem can be solved efficiently on a quantum computer. Although efficient algorithms are known for a number of special cases of this problem [3, 10–12, 14, 15, 21], the two cases known to have significant applications, the dihedral group and the symmetric group, remain unsolved. In particular, an efficient quantum algorithm for the hidden subgroup problem (HSP) over the symmetric group would lead to an efficient quantum algorithm for graph isomorphism [4, 8], and an efficient quantum algorithm for the dihedral HSP would lead to efficient quantum algorithms for certain lattice problems [24].

Although no polynomial-time algorithm is known for the dihedral HSP, Kuperberg recently discovered a subexponential-time quantum algorithm [18]. Kuperberg’s algorithm uses a superpolynomial amount of time, space, and queries; Regev subsequently improved the space requirement to be only

polynomial [25]. These algorithms are closely related to a connection between the dihedral HSP and an average case subset sum problem observed by Regev [24].

Recently, together with Bacon, we have developed an approach to the hidden subgroup problem based on the “pretty good measurement” (PGM) [2, 3]. In this approach, the PGM is used to distinguish the members of an ensemble of quantum states corresponding to the various possible hidden subgroups. For a variety of groups that can be written as the semidirect product of an abelian group and a cyclic group of prime order, we found that this measurement is closely related to a certain kind of average case algebraic problem. In particular, the measurement succeeds when the algebraic problem is likely to have a solution, and can be implemented if the solutions to that problem can be found. For the dihedral group, this problem is simply the average case subset sum problem [2]; more generally, we refer to it as the *matrix sum problem*. In some cases, the matrix sum problem can be solved, giving an efficient quantum algorithm for the corresponding hidden subgroup problem [3]. However, since the average case subset sum problem appears to be difficult, this approach has not yielded an improved algorithm for the dihedral HSP.

In this article, we show how the PGM approach provides an efficient quantum algorithm for a problem that interpolates between the abelian and dihedral hidden subgroup problems. The dihedral HSP is equivalent to the *hidden shift problem*, in which the goal is to determine a hidden shift $s \in \mathbb{Z}_N$ given two injective functions f_0, f_1 satisfying $f_0(x) = f_1(x + s)$. Instead of only two such functions, we consider M such functions, each one shifted from the previous by a fixed hidden shift s . If $M = N$, this problem is an instance of the abelian HSP on $\mathbb{Z}_N \times \mathbb{Z}_N$ with the hidden subgroup $\langle (1, s) \rangle$, which can be solved efficiently using abelian Fourier sampling. Even the case $M = N$ is classically intractable, and the problem only becomes more difficult for smaller M . In particular, for $M \ll N$, abelian Fourier sampling fails to determine the hidden shift. Using the PGM approach, we give, for any fixed integer $k \geq 3$, an efficient quantum algorithm that solves this problem for $M = \lfloor N^{1/k} \rfloor$. The algorithm works by implementing a joint measurement on k copies of certain quantum states that encode the hidden shift. Because for each $M \geq M'$ the generalized hidden shift problem on $\{0, \dots, M-1\} \times \mathbb{Z}_N$ can be reduced to the generalized hidden shift problem on $\{0, \dots, M'-1\} \times \mathbb{Z}_N$, for any fixed $\epsilon > 0$, this gives an efficient quantum algorithm for all $M \geq N^\epsilon$.

By applying the general PGM approach developed in [3], we find that the matrix sum problem corresponding to the generalized hidden shift problem is the following: given uniformly random $x \in \mathbb{Z}_N^k$ and $w \in \mathbb{Z}_N$, find $b \in \{0, \dots, M-1\}^k$ such that $b \cdot x \bmod N = w$. As this is a linear Diophantine equation with convex constraints, it is an instance of integer programming, and can be solved using Hendrik Lenstra’s algorithm for that problem [20], which is efficient as long as the dimension k is constant. Thus

our algorithm for the generalized hidden shift problem reiterates a theme of [3]: by combining abelian quantum Fourier transforms with nontrivial classical (or quantum) algorithms, one can find efficient quantum algorithms for HSP-like problems via the implementation of entangled quantum measurements. This result is encouraging since entangled measurements are known to be necessary for some hidden subgroup problems—in particular, for the symmetric group [22].

The remainder of this article is organized as follows. In Section 2, we describe the generalized hidden shift problem in detail and explain how it can be viewed as a quantum state distinguishability problem. In Section 3, we review the pretty good measurement approach to such problems, prove that this approach solves the generalized hidden shift problem when the number of states is $k \geq \lceil 1/\epsilon \rceil$, and explain how it can be implemented by solving an appropriate matrix sum problem. In Section 4, we explain how the matrix sum problem can be solved efficiently (for constant k) using Lenstra’s algorithm for integer programming. Finally, we conclude in Section 5 with a discussion of the results and some open questions.

2. The generalized hidden shift problem

It is well known that the dihedral HSP is equivalent to the *hidden shift problem*, which is defined as follows. Given two injective functions $f_0 : \mathbb{Z}_N \rightarrow S$ and $f_1 : \mathbb{Z}_N \rightarrow S$ (where S is some finite set) satisfying $f_0(x) = f_1(x + s)$ for some unknown $s \in \mathbb{Z}_N$, find s . For a proof of this equivalence, see Theorem 2 of [9] and Proposition 6.1 of [18]. For certain explicit functions of interest, such as the Legendre symbol, the hidden shift problem can be solved efficiently on a quantum computer [7]. However, for arbitrary black box functions, no efficient algorithm for the hidden shift problem is known.

A natural generalization of this problem, which we call the *generalized hidden shift problem*, is as follows. Consider a single function $f : \{0, \dots, M-1\} \times \mathbb{Z}_N \rightarrow S$ satisfying two conditions: (a) for fixed b , $f(b, x) : \mathbb{Z}_N \rightarrow S$ is injective and (b) $f(b, x) = f(b+1, x+s)$ for $b = 0, 1, \dots, M-2$ for some fixed $s \in \mathbb{Z}_N$. Given such a function, our goal is again to find the hidden shift s . For $M = 2$, this problem is simply the usual hidden shift problem (with $f_b(x) = f(b, x)$ for $b = 0, 1$), and hence is equivalent to the dihedral HSP. For $M = N$, this problem is an instance of the abelian hidden subgroup problem (where the hidden subgroup is $\langle (1, s) \rangle \leq \mathbb{Z}_N \times \mathbb{Z}_N$). As a step toward understanding the dihedral HSP, we would like to investigate the difficulty of the problem for intermediate values of M . (Note that for intermediate values of M , the generalized hidden shift problem is apparently not an instance of the HSP for any group.)

On a quantum computer, this problem can be turned into a state distinguishability problem in the same manner as the standard approach to the hidden subgroup problem. Prepare a uniform superposition over all values of $b \in \{0, \dots, M-1\}$ and $x \in \mathbb{Z}_N$ and then compute the value of $f(b, x)$,

giving the state

$$\frac{1}{\sqrt{MN}} \sum_{b=0}^{M-1} \sum_{x \in \mathbb{Z}_N} |b, x, f(b, x)\rangle. \quad (1)$$

Then measure the third register, giving the state

$$|\phi_{x,s}\rangle := \frac{1}{\sqrt{M}} \sum_{b=0}^{M-1} |b, x + bs\rangle \quad (2)$$

for some unknown $x \in \mathbb{Z}_N$. Equivalently, the result is the mixed state described by the density matrix

$$\rho_s := \frac{1}{N} \sum_{x \in \mathbb{Z}_N} |\phi_{x,s}\rangle \langle \phi_{x,s}|. \quad (3)$$

Using a single copy of the state, we can identify s by the standard period finding algorithm (e.g., as in Shor's algorithm) only when M is very large. Given the state $|\phi_{x,s}\rangle$, we can try to find s by applying the Fourier transform over $\mathbb{Z}_N \times \mathbb{Z}_N$ to the two registers, which yields the state

$$\frac{1}{N\sqrt{M}} \sum_{y,z \in \mathbb{Z}_N} \omega^{xz} \sum_{b=0}^{M-1} \omega^{b(y+sz)} |y, z\rangle, \quad (4)$$

where $\omega := \exp(2\pi i/N)$. In the case $M = N$, this state equals

$$\frac{1}{\sqrt{N}} \sum_{z \in \mathbb{Z}_N} \omega^{xz} |-sz, z\rangle. \quad (5)$$

Measuring in the computational basis, we will observe $(-sz, z)$ for a uniformly random $z \in \mathbb{Z}_N$. If z is invertible modulo N , which happens with probability $\Omega(1/\log \log N)$, then we can deduce s immediately from the values $-sz$ and z . However, in general, for $M \leq N$, the outcome will only be of the form $(-sz, z)$ with probability M/N . If $M \leq N^\epsilon$ with $\epsilon < 1$, this probability is exponentially small in $\log N$, and the approach fails. A similar argument shows that an analogous approach using a Fourier transform over $\mathbb{Z}_M \times \mathbb{Z}_N$ followed by a computational basis measurement also fails for $M \ll N$. (Note that $\text{poly}(\log N)$ such classical samples information theoretically determine the answer even for $M = 2$ [9], but it is not known how to process this data efficiently.)

Instead, we will use $k > 1$ states and apply the pretty good measurement. To see the connection to the matrix sum problem, it is helpful to write these states in a different basis. Fourier transforming the second register over \mathbb{Z}_N , we find

$$\tilde{\rho}_s^{\otimes k} = \frac{1}{(MN)^k} \sum_{x \in \mathbb{Z}_N^k} \sum_{b,c \in \{0, \dots, M-1\}^k} \omega^{(b \cdot x - c \cdot x)s} |b, x\rangle \langle c, x|. \quad (6)$$

Now introduce the states

$$|S_w^x\rangle := \frac{1}{\sqrt{\eta_w^x}} \sum_{b \in S_w^x} |b\rangle \quad (7)$$

which are uniform superpositions over the solutions of the matrix sum problem,

$$S_w^x := \{b \in \{0, \dots, M-1\}^k : b \cdot x = w \bmod N\}. \quad (8)$$

Here the number of solutions is $\eta_w^x := |S_w^x|$. If there are no solutions (i.e., if $\eta_w^x = 0$), then we define $|S_w^x\rangle := 0$. In terms of these states, we can rewrite (6) as

$$\tilde{\rho}_s^{\otimes k} = \frac{1}{(MN)^k} \sum_{x \in \mathbb{Z}_N^k} \sum_{w, v \in \mathbb{Z}_N} \omega^{(w-v)s} \sqrt{\eta_w^x \eta_v^x} |S_w^x, x\rangle \langle S_v^x, x|. \quad (9)$$

Given such a state, we would like to determine the value of s .

3. Pretty good measurement approach

In this section, we review the PGM approach to distinguishing hidden subgroup states [2, 3] as applied to the generalized hidden shift states (9).

The *pretty good measurement* (also known as the square root measurement or least squares measurement) is a measurement that often does well at distinguishing the members of an ensemble of quantum states [13] (and in fact is sometimes optimal in a certain sense). For an ensemble of states $\{\sigma_j\}$ with equal a priori probabilities, the pretty good measurement is the POVM with elements

$$E_j := \Sigma^{-1/2} \sigma_j \Sigma^{-1/2} \quad (10)$$

where

$$\Sigma := \sum_j \sigma_j \quad (11)$$

and where the inverse is taken over the support of the ensemble.

For the states (9), the PGM normalization matrix is

$$\Sigma = \frac{N}{(MN)^k} \sum_{x \in \mathbb{Z}_N^k} \sum_{w \in \mathbb{Z}_N} \eta_w^x |S_w^x, x\rangle \langle S_w^x, x|, \quad (12)$$

giving POVM elements

$$E_j = \frac{1}{N} \sum_{x \in \mathbb{Z}_N^k} \sum_{w, v \in \mathbb{Z}_N} \omega^{(w-v)j} |S_w^x, x\rangle \langle S_v^x, x|. \quad (13)$$

The probability of successfully identifying the hidden shift s is independent of s , and is given by

$$\Pr(\text{success}) := \text{tr } E_s \tilde{\rho}_s^{\otimes k} \quad (14)$$

$$= \frac{1}{M^k N^{k+1}} \sum_{x \in \mathbb{Z}_N^k} \left(\sum_{w \in \mathbb{Z}_N} \sqrt{\eta_w^x} \right)^2. \quad (15)$$

Using this expression, we can show that the success probability is appreciable when the matrix sum problem is likely to have a solution. Specifically, we have

LEMMA 1 (Lemma 2 of [3]). *If $\Pr(\eta_w^x \geq \alpha) \geq \beta$ for uniformly random $x \in \mathbb{Z}_N^k$ and $w \in \mathbb{Z}_N$ (i.e., if most instances of the matrix sum problem have many solutions), then $\alpha\beta^2 N/M^k \leq \Pr(\text{success}) \leq M^k/N$.*

PROOF. For the upper bound, we have

$$\Pr(\text{success}) \leq \frac{M}{N^{k+1}} \sum_{x \in \mathbb{Z}_N^k} \left(\sum_{w \in \mathbb{Z}_N} \eta_w^x \right)^2 = \frac{M^k}{N} \quad (16)$$

since the η 's are integers and $\sum_{w \in \mathbb{Z}_N} \eta_w^x = M^k$ for any x . For the lower bound, we have

$$\Pr(\text{success}) \geq \frac{N}{M^k} \left(\frac{1}{N^{k+1}} \sum_{x \in \mathbb{Z}_N^k} \sum_{w \in \mathbb{Z}_N} \sqrt{\eta_w^x} \right)^2 \quad (17)$$

by Cauchy's inequality applied to (15). Now

$$\frac{1}{N^{k+1}} \sum_{x \in \mathbb{Z}_N^k} \sum_{w \in \mathbb{Z}_N} \sqrt{\eta_w^x} \geq \sqrt{\alpha} \Pr(\eta_w^x \geq \alpha), \quad (18)$$

so by the hypothesis, $\Pr(\text{success}) \geq \alpha\beta^2 N/M^k$ as claimed. \square

For uniformly random $x \in \mathbb{Z}_N^k$ and $w \in \mathbb{Z}_N$, the expected number of matrix sum solutions is

$$\mu := \mathbf{E}_{x,w}[\eta_w^x] = \frac{M^k}{N}, \quad (19)$$

where we have again used the fact that $\sum_w \eta_w^x = M^k$ for any x . Thus, we expect the matrix sum problem to typically have no solutions for $k \ll \log N / \log M$, many solutions for $k \gg \log N / \log M$, and a constant number of solutions for $k \approx \log N / \log M$. This intuition can be formalized as follows:

LEMMA 2. *For the generalized hidden shift problem with $M = \lfloor N^{1/k} \rfloor$ with $k \geq 3$ and N sufficiently large, $\Pr(1 \leq \eta_w^x \leq 4)$ is lower bounded by a constant.*

A proof is given in the appendix.

Together, Lemmas 1 and 2 show that the PGM has at least a constant probability of successfully identifying the hidden shift. In fact, it turns out that the PGM is the POVM that maximizes the probability of successfully determining s given the states (9). For more details, we refer the reader to Section 4 of [2] and Section 4.4 of [3].

Finally, to give an efficient algorithm based on the PGM, we must show how the measurement can be implemented efficiently on a quantum computer. Such an implementation can be achieved using Neumark's theorem

[23], which states that any POVM can be realized by a unitary transformation U on the system together with an ancilla followed by a measurement in the computational basis. For a POVM consisting of N rank one operators $F_j = |f_j\rangle\langle f_j|$ in a D -dimensional Hilbert space, U has the block form

$$U = \begin{pmatrix} V & X \\ Y & Z \end{pmatrix} \quad (20)$$

where the rows of the $N \times D$ matrix V are the D -vectors $\langle f_j|$, i.e., $V = \sum_{j=1}^N |j\rangle\langle f_j|$. (In particular, if all the vectors $|f_j\rangle$ have unit length, as will be the case below, $X = 0$.)

Recall from (13) that the POVM operators for the PGM on hidden subgroup states can be written

$$E_j = \sum_{x \in \mathbb{Z}_N^k} E_j^x \otimes |x\rangle\langle x| \quad (21)$$

where

$$E_j^x := |e_j^x\rangle\langle e_j^x| \quad (22)$$

with

$$|e_j^x\rangle := \frac{1}{\sqrt{N^k}} \sum_{w \in \mathbb{Z}_N} \omega^{wj} |S_w^x\rangle. \quad (23)$$

In other words, each E_j is block diagonal, with blocks labeled by $x \in \mathbb{Z}_N^k$, and where each block is rank one. Thus, the measurement can be implemented in a straightforward way by first measuring the block label x and then performing the POVM $\{E_j^x\}_{j \in \mathbb{Z}_N}$ conditional on the first measurement result.

To implement the POVM $\{E_j^x\}_{j \in \mathbb{Z}_N}$ using Neumark's theorem, we would like to implement the unitary transformation U^x with the upper left submatrix

$$V^x = \frac{1}{\sqrt{N}} \sum_{j, w \in \mathbb{Z}_N} \omega^{-wj} |j\rangle\langle S_w^x|. \quad (24)$$

It is convenient to perform a Fourier transform (over \mathbb{Z}_N) on the left (i.e., on the index j), giving a unitary operator \tilde{U}^x with upper left submatrix

$$\tilde{V}^x = \frac{1}{N} \sum_{j, w, v \in \mathbb{Z}_N} \omega^{(w-v)j} |w\rangle\langle S_v^x| \quad (25)$$

$$= \sum_{w \in \mathbb{Z}_N} |w\rangle\langle S_w^x|. \quad (26)$$

Therefore, the PGM can be implemented efficiently if we can efficiently perform a unitary transformation satisfying

$$|w, x\rangle \mapsto |S_w^x, x\rangle \quad (27)$$

for all matrix sum instances (x, w) with $\eta_w^x > 0$. We refer to (27) as *quantum sampling* of solutions to the matrix sum problem. If we can efficiently quantum sample from matrix sum solutions, then by running the circuit in

reverse, we can efficiently implement \tilde{U}^x , and hence the desired measurement.

By applying these unitary transformations directly to the state (9), we can obtain a description of the algorithm without reference to generalized measurement. Performing the inverse of the quantum sampling transformation (27) followed by a Fourier transform, we obtain the state

$$\rho' := \frac{1}{N^k} \sum_{x \in \mathbb{Z}_N^k} |\rho'_x, x\rangle \langle \rho'_x, x| \quad (28)$$

where

$$|\rho'_x\rangle := \frac{1}{\sqrt{M^k N}} \sum_{j, w \in \mathbb{Z}_N} \omega^{w(s-j)} \sqrt{\eta_w^x} |j\rangle. \quad (29)$$

Roughly speaking, if the distribution of η_w^x is close to uniform, then the sum over w in (29) is close to zero unless $j = s$, so that a measurement of the first register is likely to yield the hidden shift s .

In general, it may be difficult to implement (27) exactly. Instead, we may only be able to perform an approximate quantum sampling transformation satisfying

$$|w, x\rangle \mapsto \begin{cases} |S_w^x, x\rangle & (x, w) \in Z_{\text{good}} \\ |\mu_w^x, x\rangle & (x, w) \in Z_{\text{bad}} \end{cases} \quad (30)$$

for some states $|\mu_w^x\rangle$, where $Z_{\text{good}}, Z_{\text{bad}}$ form a partition of the matrix sum instances (x, w) for which $\eta_w^x > 0$. The good matrix sum problem instances $(x, w) \in Z_{\text{good}}$ are those for which the quantum sampling can be done correctly. Assuming the bad matrix sum instances $(x, w) \in Z_{\text{bad}}$ can be recognized, we can ensure that $\langle S_w^x | \mu_{w'}^x \rangle = 0$ for all $x \in \mathbb{Z}_N^k, w, w' \in \mathbb{Z}_N$. Applying the approximate quantum sampling transformation followed by the Fourier transform gives the state

$$\rho'_{\text{apx}} = \frac{1}{N^k} \sum_{x \in \mathbb{Z}_N^k} |\rho'_{x, \text{apx}}, x\rangle \langle \rho'_{x, \text{apx}}, x| \quad (31)$$

where

$$|\rho'_{x, \text{apx}}\rangle := \frac{1}{\sqrt{M^k N}} \sum_{j \in \mathbb{Z}_N} \left(\sum_{(x, w) \in Z_{\text{good}}} \omega^{w(s-j)} \sqrt{\eta_w^x} |j\rangle + \sum_{(x, w) \in Z_{\text{bad}}} \omega^{w(s-j)} \sqrt{\eta_w^x} |\nu_j^x\rangle \right) \quad (32)$$

for some states $|\nu_j^x\rangle$ with $\langle j | \nu_{j'}^x \rangle = 0$ for all $x \in \mathbb{Z}_N^k, j, j' \in \mathbb{Z}_N$ (since $\langle S_w^x | \mu_{w'}^x \rangle = 0$ for all $x \in \mathbb{Z}_N^k, w, w' \in \mathbb{Z}_N$ and (30) is unitary). The fidelity between the ideal final state ρ' and the actual final state ρ'_{apx} resulting from

approximate quantum sampling is thus

$$\frac{1}{(MN)^k} \sum_{(x,w) \in Z_{\text{good}}} \eta_w^x. \quad (33)$$

Now $\eta_w^x > 1$ for all $(x, w) \in Z_{\text{good}}$, so if $|Z_{\text{good}}|$ is sufficiently large, the actual final state is close to the ideal final state, and hence a measurement of the first register yields the hidden shift s with reasonable probability. As we will show in the next section, the instances with $1 \leq \eta_w^x \leq 4$ can be quantum sampled efficiently (i.e., these instances are good). Then, letting $M = \lfloor N^{1/k} \rfloor$, Lemma 2 shows that $|Z_{\text{good}}|/N^{k+1}$ is lower bounded by a constant, and thus the fidelity between ρ' and ρ'_{apx} is lower bounded by a constant, so that the probability of successfully determining s is lower bounded by a constant.

4. Solution of the matrix sum problem

Recall that the matrix sum problem for the generalized hidden shift problem is the following: given $x \in \mathbb{Z}_N^k$ and $w \in \mathbb{Z}_N$ chosen uniformly at random, find $b \in \{0, \dots, M-1\}^k$ such that $b \cdot x = w \pmod N$. This is a linear equation over \mathbb{Z}_N in k variables, where the solutions are required to come from $\{0, \dots, M-1\}$. Such solutions can be found using integer programming, which has an efficient algorithm if M is sufficiently large. We assume $M = \lfloor N^{1/c} \rfloor$ for some positive integer $c \geq 3$. Since we can always decrease M by only considering a subset of the inputs to the first argument of the hiding function f , this will not constitute a loss of generality. According to Lemma 2, we take $k = c$ so that there are between 1 and 4 solutions with probability at least some constant.

To see the connection to integer programming, we note that the solutions form an integer lattice. We begin by considering the equation $b \cdot x = w \pmod N$ as a $(k+1)$ -variable linear equation over all the integers \mathbb{Z} . Define an extension of x by $\bar{x} := (x_1, \dots, x_k, N)$ and consider the solutions $\bar{b} \in \mathbb{Z}^{k+1}$ of the equation $\bar{b} \cdot \bar{x} = w$. For any $b \in \mathbb{Z}^k$ that solves the equation $b \cdot x = w \pmod N$, there is a unique $\lambda \in \mathbb{Z}$ such that $\bar{b} = (b, \lambda)$ is a solution to $\bar{b} \cdot \bar{x} = w$; and conversely, for any $\bar{b} \in \mathbb{Z}^{k+1}$ that solves $\bar{b} \cdot \bar{x} = w$, there is a unique $b \in \mathbb{Z}^k$ (namely, the first k components of \bar{b}) that solves $b \cdot x = w \pmod N$. Hence there is a bijection between the solutions $\bar{b} \in \mathbb{Z}^{k+1}$ to the equation $\bar{b} \cdot \bar{x} = w$ and the solutions $b \in \mathbb{Z}^k$ to the equation $b \cdot x = w \pmod N$.

By Lemma 4 in the appendix, we see that the linear Diophantine equation $\bar{b} \cdot \bar{x} = w$ will have no solutions if $\gcd(x_1, \dots, x_k, N)$ does not divide w . If $\bar{b} \cdot \bar{x} = w$ does have a solution, then the solutions \bar{b} comprise a shifted k -dimensional lattice $\bar{b}^{(0)} + L$ with some particular solution $\bar{b}^{(0)}$ satisfying $\bar{b}^{(0)} \cdot \bar{x} = w$ and the elements of $L \subset \mathbb{Z}^{k+1}$ the solutions of the equation $\bar{b} \cdot \bar{x} = 0$. By omitting the last coordinate of these solutions, we obtain all

solutions $b \in \mathbb{Z}^k$, which comprise a shifted k -dimensional lattice in \mathbb{Z}^k :

$$b = b^{(0)} + \sum_{j=1}^k \beta_j b^{(j)} \quad (34)$$

for all $\beta_1, \dots, \beta_k \in \mathbb{Z}$. Due to the aforementioned bijection, each solution $b \in \mathbb{Z}^k$ has a unique set of coordinates $\beta_1, \dots, \beta_k \in \mathbb{Z}$. The vectors $b^{(0)}, \dots, b^{(k)} \in \mathbb{Z}^k$ can be found efficiently by applying the extended Euclidean algorithm to the equation $\bar{b} \cdot \bar{x} = w$ (see for example Algorithm 1.3.6 in [6]).

To solve the matrix sum problem, we would like to find the solutions b that lie in $\{0, \dots, M-1\}^k$, which is the set of integer points in the convex region described by the inequalities

$$0 \leq b_i \leq M-1, \quad i = 1, \dots, k. \quad (35)$$

The problem of finding such points (or more precisely, deciding whether such a point exists) is simply an instance of integer programming in k dimensions, which can be solved efficiently if k is a constant. In general, the integer programming problem is the following. Given a rational matrix $A \in \mathbb{Q}^{m \times k}$ and a rational vector $\gamma \in \mathbb{Q}^m$, does there exist an integral vector $\beta \in \mathbb{Z}^k$ such that $A\beta \leq \gamma$? Although this general problem is NP-complete [5, 17], if the dimension k is held constant, then the problem can be solved in time polynomial in the input size [20] using an algorithm based on lattice basis reduction [19].

By rewriting the convex constraints (35) in terms of the lattice of solutions (34), we see that solutions of the matrix sum problem correspond precisely to vectors $\beta \in \mathbb{Z}^k$ satisfying the constraints

$$\sum_{j=1}^k \beta_j b_i^{(j)} \leq (M-1) - b_i^{(0)}, \quad i = 1, \dots, k \quad (36)$$

$$-\sum_{j=1}^k \beta_j b_i^{(j)} \leq b_i^{(0)}, \quad i = k+1, \dots, 2k. \quad (37)$$

But this is precisely an instance of integer programming in k dimensions with $m = 2k$ constraints, with

$$A_{ij} = \begin{cases} b_i^{(j)} & i = 1, \dots, k \\ -b_{i-k}^{(j)} & i = k+1, \dots, 2k \end{cases} \quad (38)$$

$$\gamma_i = \begin{cases} (M-1) - b_i^{(0)} & i = 1, \dots, k \\ b_{i-k}^{(0)} & i = k+1, \dots, 2k. \end{cases} \quad (39)$$

Therefore, it can be solved efficiently whenever k is a constant. Note that integer programming as described above is a decision problem, whereas we would like to find the actual solutions. However, this is easily accomplished using bisection, recursively dividing the set $\{0, \dots, M-1\}^k$ into halves, to find all of the solutions efficiently (for the cases in which there are few

solutions—in particular, for those for which there are between 1 and 4 solutions).

Overall, we see that we can efficiently solve the matrix sum problem (in a regime where the pretty good measurement solves the generalized hidden shift problem with constant probability) whenever $M \geq N^\epsilon$ for some fixed $\epsilon > 0$. For the cases in which the number of solutions is small, they can be explicitly enumerated, and hence we can efficiently perform the approximate quantum sampling transformation (30) (see for example footnote 2 of [3]). Therefore, we find the following result.

THEOREM 3. *The generalized hidden shift problem with $M \geq N^\epsilon$ for any fixed $\epsilon > 0$ can be solved in time $\text{poly}(\log N)$ on a quantum computer.*

PROOF. Given ϵ , we will use $k = \max\{\lceil 1/\epsilon \rceil, 3\}$ copies of the unknown quantum state (3). Because the generalized hidden shift problem on the full domain $\{0, \dots, M-1\} \times \mathbb{Z}_N$ can be solved by solving the same problem on a reduced domain $\{0, \dots, M'-1\} \times \mathbb{Z}_N$ with $M' \leq M$, it is sufficient to prove the theorem for a specific $M \leq N^\epsilon$. We will do this for $M = \lfloor N^{1/k} \rfloor$.

By Lemma 2, there is a constant probability of having between 1 and 4 solutions to the random matrix sum equation $b \cdot x = w \bmod N$, and hence the success probability of the PGM is also a constant (by Lemma 1). The efficient approximate implementation of this measurement is described in the second half of Section 3 in combination with the results on integer programming that are described in the first part of Section 4. \square

In fact, the algorithm remains efficient even if ϵ decreases (very) slowly with N . Lenstra’s algorithm for integer programming in dimension k takes time $2^{O(k^3)}$ [20], so the generalized hidden shift problem can be solved efficiently for $M = N^{O(1/(\log \log N)^{1/3})}$. Indeed, a subsequent improvement by Kannan solves k -dimensional integer programming in time $2^{O(k \log k)}$ [16], which can be used to decrease M slightly further.

5. Discussion

We have applied the PGM approach to the generalized hidden shift problem, which interpolates from the dihedral HSP to the abelian HSP as M varies from 2 to N . We found an efficient quantum algorithm for this problem for any $M \geq N^\epsilon$ with ϵ fixed (or decreasing very slowly with N). The algorithm works by solving the matrix sum problem using Lenstra’s algorithm for integer programming in constant dimensions, thereby illustrating (as in [3]) that nontrivial classical algorithms can be useful for implementing entangled measurements to distinguish states obtained by Fourier sampling.

Our original motivation for studying this problem was the observation that a solution to the generalized hidden shift problem for sufficiently small M could lead to new algorithms for the unique shortest vector in a lattice problem, just as Regev showed for the case $M = 2$ [24]. Unfortunately,

$M \geq N^\epsilon$ does not appear to be sufficiently small to yield interesting lattice algorithms. Nevertheless, attempting to solve the generalized hidden shift problem for yet smaller M may be a promising path toward improved quantum algorithms for lattice problems. Indeed, for the case $M = 2$, Kuperberg's subexponential-time algorithm outperforms the algorithm given in this paper, so it seems likely that an improved algorithm could be found for values of M intermediate between 2 and N^ϵ .

Another problem suggested by this work is the following generalization of graph isomorphism. Suppose that we are given a list of n -vertex graphs G_0, \dots, G_{M-1} , and are promised that either no two graphs are isomorphic, or $G_b = \pi(G_{b+1})$ for some fixed permutation $\pi \in S_n$ for $b = 0, 1, \dots, M-2$. It would be interesting to show that this problem can be solved efficiently even for very large M (where the graphs can be specified by a black box in the case where M is superpolynomial in n).

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Appendix: Number of solutions of the matrix sum problem

In this section, we prove Lemma 2. Before giving the proof, we need the following fact:

LEMMA 4. *For any fixed b , the number of solutions $x \in \mathbb{Z}_N^k$ to the equation $b \cdot x = 0 \pmod{N}$ is $N^{k-1} \gcd(b_1, \dots, b_k, N)$.*

PROOF. First, consider the case where $N = p^r$ is a prime power. Then $\gcd(b_1, \dots, b_k, p^r) = p^s$ for some $s \in \{0, 1, \dots, r\}$. In particular, there must be an index i such that $\gcd(b_i, p^r) = p^s$, and hence $b_i = cp^s$ for some $c \in \{1, \dots, p-1\}$. Without loss of generality, assume $i = 1$. Now we can rewrite the equation $b \cdot x = 0$ as $cp^s x_1 + \sum_{j=2}^k b_j x_j = 0 \pmod{p^r}$, or equivalently, since p^s is a common divisor of all b_j , $cx_1 = -\sum_{j=2}^k b'_j x_j \pmod{p^{r-s}}$ where $b'_j = b_j/p^s$. Because $c \in \mathbb{Z}_{p^r}^\times$, for any fixed $(x_2, \dots, x_k) \in \mathbb{Z}_{p^r}^{k-1}$, there are p^s solutions $x_1 = (\sum_{j=2}^k b'_j x_j)/c + \lambda p^{r-s} \pmod{p^r}$ (one solution for each $\lambda \in \{0, \dots, p^s - 1\}$). Hence the total number of solutions (x_1, \dots, x_k) is $N^{k-1} p^s$, proving the lemma for the case $N = p^r$.

Now if N is not a prime power, let $N = p_1^{r_1} \cdots p_t^{r_t}$ be the factorization of N into powers of distinct primes, and let $\tau : \mathbb{Z}_N \rightarrow \mathbb{Z}_{p_1^{r_1}} \times \cdots \times \mathbb{Z}_{p_t^{r_t}}$ be the ring isomorphism provided by the Chinese remainder theorem: for $x \in \mathbb{Z}_N$, $\tau(x) = (x \pmod{p_1^{r_1}}, \dots, x \pmod{p_t^{r_t}})$. Since τ is a ring isomorphism, $b \cdot x = 0 \pmod{N}$ if and only if $b \cdot \tau(x)_i = 0 \pmod{p_i^{r_i}}$ for all $i = 1, \dots, t$. By

the special case of the lemma for N a prime power, the number of solutions to the i th equation is $p_i^{r_i(k-1)} \gcd(b_1, \dots, b_k, p_i^{r_i})$; hence the total number of solutions is $\prod_{i=1}^t p_i^{r_i(k-1)} \gcd(b_1, \dots, b_k, p_i^{r_i}) = N^{k-1} \gcd(b_1, \dots, b_k, N)$ as claimed. \square

Now we are ready to give the proof of Lemma 2:

LEMMA 2. *For the generalized hidden shift problem with $M = \lfloor N^{1/k} \rfloor$ with $k \geq 3$ and N sufficiently large, $\Pr(1 \leq \eta_w^x \leq 4)$ is lower bounded by a constant.*

PROOF. The main idea of the proof is the same as in the proof of Lemma 5 of [3]: we show that the variance of η_w^x is small, so that the number of solutions of the matrix sum problem is typically close to its mean. Because we have $M = \lfloor N^{1/k} \rfloor$, the mean value of η_w^x is $\mu := \mathbf{E}_{x,w}[\eta_w^x] = M^k/N = 1 + O(1/N)$ as N grows.

The variance of the number of solutions $b \in \{0, \dots, M-1\}^k$ of the equation $b \cdot x = w \bmod N$ for uniformly random $x \in \mathbb{Z}_N^k, w \in \mathbb{Z}_N$ is $\sigma^2 := \mathbf{E}_{x,w}[(\eta_w^x)^2] - \mu^2$, and

$$\mathbf{E}_{x,w}[(\eta_w^x)^2] = \frac{1}{N^{k+1}} \sum_{x \in \mathbb{Z}_N^k, w \in \mathbb{Z}_N} (\eta_w^x)^2 \quad (40)$$

$$= \frac{1}{N^{k+1}} \sum_{x \in \mathbb{Z}_N^k, w \in \mathbb{Z}_N} \left(\sum_b \delta_{b \cdot x, w} \right) \left(\sum_c \delta_{c \cdot x, w} \right) \quad (41)$$

$$= \frac{1}{N^{k+1}} \sum_{x,w} \left(\sum_b \delta_{b \cdot x, w} + \sum_{b \neq c} \delta_{b \cdot x, w} \delta_{c \cdot x, w} \right) \quad (42)$$

(with the b, c summations over $\{0, \dots, M-1\}^k$). The first (diagonal) term is just the mean. To handle the second (off-diagonal) term, we can write

$$\mathbf{E}_{x,w}[(\eta_w^x)^2] = \mu + \frac{1}{N^{k+1}} \sum_{b \neq c} \sum_{x \in \mathbb{Z}_N^k} \delta_{b \cdot x, c \cdot x} \sum_{w \in \mathbb{Z}_N} \delta_{b \cdot x, w} \quad (43)$$

$$= \mu + \frac{1}{N^{k+1}} \sum_{b \neq c} \sum_{x \in \mathbb{Z}_N^k} \delta_{b \cdot x, c \cdot x} \quad (44)$$

$$= \mu + \frac{1}{N^2} \sum_{b \neq c} \gcd(b_1 - c_1, \dots, b_k - c_k, N) \quad (45)$$

$$\leq \mu + \frac{1}{N^2} \sum_{b \neq c} \gcd(b_1 - c_1, \dots, b_k - c_k) \quad (46)$$

where the next to last step follows from Lemma 4. Now for any $q \in \{0, \dots, M-1\}$, for a fixed value of c_i , there are $1 + \lfloor (M - c_i - 1)/q \rfloor + \lfloor c_i/q \rfloor \leq 1 + M/q$ choices of $b_i \in \{0, \dots, M-1\}$ that are divisible by q , and hence the number of b, c such that $\gcd(b_1 - c_1, \dots, b_k - c_k) = q$ is upper bounded by $M^k [(M + q)/q]^k$. Therefore, for fixed $k \geq 3$ and $M^k/N = 1 + O(1/N)$, we have

$$\mathbf{E}_{x,w} [(\eta_w^x)^2] \leq \mu + \frac{M^k}{N^2} \sum_{q=1}^{M-1} q \left(\frac{M+q}{q} \right)^k \quad (47)$$

$$= \mu + \frac{M^k}{N^2} \sum_{q=1}^{M-1} \sum_{j=0}^k \binom{k}{j} \frac{M^j}{q^{j-1}} \quad (48)$$

$$\leq \mu + \frac{M^k}{N^2} \left[O(M^2 \log M) + \sum_{j=3}^k \binom{k}{j} M^j \sum_{q=1}^{\infty} \frac{1}{q^{j-1}} \right] \quad (49)$$

$$\leq \mu + \frac{M^k}{N^2} \left[O(M^2 \log M) + \frac{\pi^2}{6} \sum_{j=3}^k \binom{k}{j} M^j \right] \quad (50)$$

$$= \mu + \frac{\pi^2}{6} + o(1). \quad (51)$$

where in the next to last step we have used the fact that $\sum_{q=1}^{\infty} q^{-(j-1)} \leq \pi^2/6$ for any $j \geq 3$. As $\mu = M^k/N = 1 + o(1)$, we find $\sigma^2 = \mathbf{E}_{x,w} [(\eta_w^x)^2] - \mu^2 \leq \pi^2/6 + o(1)$.

Since the variance is small, Chebyshev's inequality shows that the probability of deviating far from the mean number of solutions is small:

$$\Pr(|\eta_w^x - \mu| \geq \Delta) \leq \frac{\sigma^2}{\Delta^2}. \quad (52)$$

Putting $\Delta = 4$ and using the fact that η_w^x must be an integer, we find $\Pr(\eta_w^x \geq 5) \leq \pi^2/96 + o(1)$.

To see that we are unlikely to have no solutions, we need a slightly stronger bound than the Chebyshev inequality. Since $\eta_w^x \in \mathbb{N}$, we have $\Pr(\eta_w^x = 0) \leq \sigma^2/(\mu^2 + \sigma^2)$ [1, p. 58]. Now, noting that the gcd in (45) is at least 1, we have

$$\mathbf{E}_{x,w} [(\eta_w^x)^2] \geq \mu + \frac{M^k(M^k - 1)}{N^2} = 2 + o(1) \quad (53)$$

so that $\sigma^2 \geq 1 + o(1)$. Therefore, we find $\Pr(\eta_w^x = 0) \leq \pi^2/12 + o(1)$. Combining these results, we see that $\Pr(1 \leq \eta_w^x \leq 4) \geq 1 - 3\pi^2/32 + o(1) \geq 0.0747 + o(1)$, so that the probability is lower bounded by a constant for sufficiently large N . \square

While the above bounds apply to arbitrary values of N , they are not tight, and better bounds can be obtained using knowledge of the factorization of N . For example, if N is prime, $\sigma^2 \sim 1$. For $k = 2$, the above

argument is not sufficient except for special values of N (such as N prime); indeed, if N has an unbounded number of distinct prime factors, then it appears that the variance of η_w^x might be unbounded. However, for this case, one can simply decrease M and use $k = 3$ copies, as mentioned previously.

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